Application of System of Linear Equations and Gauss-Jordan Elimination to Environmental Science in a Classroom Setting

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Abstract:
System of linear equation arise in various situations and Environmental Science is not an exceptions. This is also one of the first few topics that is covered in any standard linear algebra course. Thus this modulo can be implanted fairly early in the semester just after finishing Gauss-Jordan elimination methods are taught. By implementing this modulo not only students will be motivated to see a real environmental application of linear algebra but also connect theory with application early in the semester while typically application is usually taught toward the end or middle of the semester. Students will start with a warm-up example and will gradually be introduced the various parts of this modulo. Several types of questions are asked throughout this modulo which will guide the students to understand the concepts. At the end of the module, a project is provided as an expansion and extension of this modulo which can be assigned as a short project or as a classroom activity. As a byproduct of this project, student will learn the “Mathematica” software, a very user friendly computer algebra system (CAS).

Keywords:
Engineering; Education; Linear Algebra; Modulo; STEM (Science Technology Engineering Mathematics)

1. INTRODUCTION

Connecting theory and application is a challenging but important problem. This is important for all students, but particularly important for students majoring in STEM education. We need to motivate our engineering students so they can be successful in their educational and occupational lives. As we see from many years of experience of teaching Mathematics and other STEM related disciplines that motivating, by nature, is not an easy task. When it comes to STEM education, this becomes an even more difficult task. This, in part, probably because in a STEM related discipline, the students are required to give more continuous attention and effort to understand the difficult concepts. On top of this, the groups of students that we are working on are, for most part, full time workers with family responsibilities. Most of them are minority students and have many other social, economic, and political problems to deal with in their personal and professional lives. This is especially true for students in the evening classes, who after a long day of work, have difficulty concentrating in class and, even when they understand the lecture, difficulty retaining the knowledge and manipulating it in the future (especially during an exam). In fact, one student from our calculus I class made the following comment:
“It is really difficult for me to keep my eyes open, and keeping concentration after the first 20 minutes of the lecture is almost impossible for me. Gradually, as the semester goes on, the classroom becomes my bedroom.”

The comment above aligns well with research findings. McKeachie points out the following:

In a typical 50-minute lecture class, students retain 70% of what is conveyed in the first 10 minutes but only 20% from the last 10 minutes. If we really want to get our message across, we need to orchestrate “the material” in a multi-faceted way across the range of student learning style. (McKeachie, 1994)

Similar comments can be found on [1–5]. So creating modules provide one way to connect boring theory with exited application and create and entice the interest of students. This module, in particular, is an application of System of linear equation which can arise in various application, including the followings:

1. A natural mathematical models of various real life application
2. An approximation to a non-linear model
3. A step of solving other mathematical problems including ordinary and partial differential equations

This modulo introduce an example of a first kind – system of linear equations as a model of real life problems – and our real life problem will be related to Environmental Science.

2. STUDENT LEARNING OUTCOMES

1. Learn application of system of linear equations to determine average caloric value of specific fishes
2. Analyze solutions and extract conclusion
3. Learn sensitivity analysis of system of linear equations
4. Learn to use “Mathematica” to solve system of linear equations
5. Differentiate between “Well-determined” and “Ill-determined” linear systems
6. Numerical solutions including “Partial pivoting” and “Gauss-Seidel Iteration”

3. THEORETICAL BACKGROUND

The students will be taught the following topics before this modulo is implemented in the classroom:

1. System of linear equation
2. Solving system of linear equation using Gauss-Jordan Elimination
3. Reduced row echelon form and analyze various types of solutions to system of linear equations

The following topics will be covered as part of this modulo and is not required to be taught before the implementation but will be helpful if someone wants to introduce these ideas to students:

1. Numerical solutions to system of linear equations including “partial pivoting”, “Jacobi iteration” and “Gauss-Seidel iteration”.
2. “Ill-contained system” and “well-contained system”.

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4. WARM-UP EXAMPLE MEASURING TEMPERATURE IS BIRD’S NEST

Let say you want to study the behavior of temperature in a bird’s nest. You borrowed an old instrument that uses thermistors to sense temperature. The manual of this instrument tells you that the resistance rises nearly linearly with the temperature. This linear relationship between the temperature and the resistance is well behaves in the range of 0-40 degree Celsius. Also you should note that you need to calibrate the instrument in every six months or so to account for any aging of components. The output will be resistance given in KO (Kilohms) and temperature will be measured in degree C (Celsius).

In a winter morning, say at January 7, when the temperature is 0 degree Celsius, you went to a bird’s nest (assuming the bird is outside the nest collecting food) and using the instrument you measure the resistance to be 9.8 KO. On the next day, when the temperature is little warmer, say 25 degree C (after all the weather is very unpredictable especially in New York), you perform another measurement and found the resistance to be 15.79 KO.

1. If \((r,t)\) denote the pair resistance and the corresponding temperature then write down the two pairs that is given in this question.
2. Plot the two points on the graphing paper and draw a line through the two points. Why are we drawing a line through these two points? Why not a curve line?
3. Find the slope of the line and the equation of the line.
4. We know that equation of any line has the form \(y=mx+b\) where “\(m\)” is the slope and “\(b\)” is the \(y\)-intercept. Substitute the two given points and write the corresponding two equations with two unknowns (your unknowns are being “\(m\)” and “\(b\)”). Solve the system.
5. On January 15, you measure the resistance to be 12.65 KO. What will be the approximate temperature of the bird’s nest? Do you think this is an exact measurement of the temperature of the bird’s nest?
6. If the temperature of the bird’s nest is 30 degree Celsius, what will be the resistance?
7. On a hot summer day when the temperature is nearly 50 degree Celsius, can you apply the previous method to measure the temperature of the bird’s nest? Explain.
8. On December 20, you apply the previous method to approximately find the temperature of the bird’s nest. What is the first thing that you need to do?
9. What are some of the factors that can influence your measurement?

5. THE TYPICAL EXAMPLE AND INTRODUCTION TO THE PROBLEM

After the warm-up example and some discussion about it, students are ready for the actual problems. We will start with the following problems where we have one species of fish population and they are feeding on two species of insects. The problem is to determine the average caloric value a fish is getting from a particular species of insect.

Let say in a lake there are a lot of trout and they are feeding on moths and midges. We want to determine how much caloric value a trout is getting from moths and midges. This is a difficult task. To make things less complicated, we will assume that all the trout in the lake only feeding on these two types of insects.

To start the process, we can catch some trout and examine the content in the stomach. We can count the head capsule of each insects (moths and midges) to determine how many moths and midges the trout had eaten. Then we
will use a calorimeter to measure the total caloric value of the stomach’s content. Suppose we do this for two fish and find the following data:

<table>
<thead>
<tr>
<th>Fish</th>
<th>Number of Moths</th>
<th>Number of Midges</th>
<th>Total caloric value of the stomach</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>12</td>
<td>18</td>
<td>660</td>
</tr>
<tr>
<td>B</td>
<td>8</td>
<td>14</td>
<td>480</td>
</tr>
</tbody>
</table>

Let $x_1$ represent the average caloric content of midges eaten and $x_2$ represent the average caloric content of moths eaten. The above table can be converted to the following two equations and two unknowns:

$$18x_1 + 12x_2 = 660$$  
$$14x_1 + 8x_2 = 480$$

We should always make sure we have the correct unit. $x_1$ is cal/midges and $x_2$ cal/moths.

A quick dimensional analysis checks out:

$$(\text{midges}) \times (\text{cal/midges}) + (\text{moths}) \times (\text{cal/moths}) = \text{cal}$$

Now solve these two equations for $x_1$ and $x_2$. Students should be given some time to solve it. The solution is $x_1 = 20$ cal/midges and $x_2 = 25$ cal/midges.

**Question 1:** We happen to get an integer solution for both $x_1$ and $x_2$. Do we must have to have an integer solution? Is it possible to have decimal answer as our solution? In other word, is fraction or decimal answer make sense in our situation?

In the previous calculation, we only include the measurement taken from two fishes. For any two fishes, we will have two equations and we can solve them. So for example, if we take another two fishes, we might get a little different measurement and thus our values for $x_1$ and $x_2$ will also change. One way to remedy the situation is to find several possible values of $x_1$ and $x_2$ and then take the average. This average might be a good approximation of the actual answer. So let us say you caught four more trout and take the measurement and record this in the following table:

<table>
<thead>
<tr>
<th>Fish</th>
<th>Number of Moths</th>
<th>Number of Midges</th>
<th>Total caloric value of the stomach</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>23</td>
<td>16</td>
<td>890</td>
</tr>
<tr>
<td>D</td>
<td>15</td>
<td>8</td>
<td>540</td>
</tr>
<tr>
<td>E</td>
<td>5</td>
<td>9</td>
<td>295</td>
</tr>
<tr>
<td>F</td>
<td>17</td>
<td>14</td>
<td>730</td>
</tr>
</tbody>
</table>

**Question 2:** Write down the system of equations from the fish C and D and solve for $x_1$ and $x_2$. Do the same for the fish E and F. So now you have three solutions for $x_1$ and three solutions for $x_2$. Find the average value of $x_1$ and $x_2$. This value is most likely a better approximate to the actual solution.

**Question 3:** Why do you think we are getting different measurements for different fish? What are some of the factors that may contribute to the differences of the measurement?

After students give some feedback on question 3, we can continue:

**6. SENSITIVITY ANALYSIS**

One reason that one could get different measurement for different fish has to do with different sizes of moths and midges. Some moths are bigger in size that the other months. So the bigger moths will supply more caloric value that the smaller moths. Same is true for midges. Another things can contribute to the problem is that in real life trout are
not just feeding on moths and midges. Potentially they have other foods and this also contribute to the total caloric value in the stomach. But for simplicity, we assume our trout population only feed on moths and midges.

What other factors can contribute to the different measurement? When we calculate the total caloric value of the stomach by a calorimeter, this measurement is never exact. It is very hard to get a very accurate measurement of the caloric value. So there is always a chance of errors in this measurement. So naturally we are interested to know how sensitive our final answer is going to be on the potential errors of the measurement of the total caloric value. So this bring us to the sensitivity analysis.

Let us say that we believe that there is a chance of 10% of error when we measure the caloric value. We are going back to our first table (fish A and B) and record this possible errors for each fish. So for fish A the original measurement of total caloric value of the stomach’s content is 660. But for a 10% error, we will incorporate 660 ± 10% of 660. That is the highest possible value 660 + (0.1)(660) = 726 and the lowest possible value 660 - (0.1)(660) = 594. Similarly the highest and lowest possible values for fish B are 528 and 432 respectively. We put all this information in the following table.

<table>
<thead>
<tr>
<th>Fish</th>
<th>Number of Moths</th>
<th>Number of Midge</th>
<th>Original Total caloric value of the stomach</th>
<th>A large B small</th>
<th>A small B large</th>
<th>A large B large</th>
<th>A small B large</th>
<th>Check sum row</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>12</td>
<td>18</td>
<td>660</td>
<td>726</td>
<td>594</td>
<td>726</td>
<td>594</td>
<td>3330</td>
</tr>
<tr>
<td>B</td>
<td>8</td>
<td>14</td>
<td>480</td>
<td>432</td>
<td>328</td>
<td>432</td>
<td>328</td>
<td>2422</td>
</tr>
</tbody>
</table>

Note that we incorporate all four possibilities – measurement of fish A is maximum and the measurement of fish B is minimum etc. We also add a final column of check sum row which is just the sum of each row. This is just to make sure our calculation is correct during any steps of the Gauss Jordan elimination. This last column is optional and can be omitted.

7. USE OF MATHEMATICA TO SOLVE SYSTEM OF LINEAR EQUATIONS

To download “Mathematica” and to learn more about the software, please see [6] and [7]. Now refer to the data given in the above table. We will start with the following question:

**Question 4:** The corresponding augmented matrix is:

<table>
<thead>
<tr>
<th>12</th>
<th>18</th>
<th>660</th>
<th>726</th>
<th>594</th>
<th>726</th>
<th>594</th>
<th>3330</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>8</td>
<td>480</td>
<td>432</td>
<td>432</td>
<td>528</td>
<td>528</td>
<td>2422</td>
</tr>
</tbody>
</table>

Perform the Gauss-Jordan elimination and find out the corresponding reduced row echelon form of the matrix. Use calculator if necessary for the computation.

The powerful “mathematica” software can be used to perform this type of calculation. Mathematica has a build-in command which will perform reduced row echelon form of any matrix in a second. In put the above matrix into mathematica using the following command:

\[ A = \{\{12,18,660,726,594,726,594,3330\},\{14,8,480,432,432,528,528,2422\}\} \]

Note that we denote this matrix by A.

Now to perform the Gauss-Jordan elimination through mathematica just enter the command:

\[ \text{RowReduce}[A]/\text{MatrixForm} \]

The last command “//MatrixForm” convert the list into familiar matrix form for easy visualization.
**Question 5:** Use Mathematica to check your solution for the warm-up exercise and question 2.

### 8. ANALYZE SOLUTIONS  |  ILL DETERMINED SYSTEM VS. WELL DETERMINED SYSTEM

Let us analyze the reduced row echelon form that we get in the previous example:

The five solutions for the five different output are given below:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>20</td>
<td>-26</td>
<td>18</td>
<td>22</td>
</tr>
<tr>
<td>$x_2$</td>
<td>25</td>
<td>99.5</td>
<td>22.5</td>
<td>27.5</td>
</tr>
</tbody>
</table>

First note that we did get some negative answer and the caloric value cannot be negative. So these are called erroneous solutions. The main problem to this particular situation that there is no "good way" to measure caloric value. And one of the conclusion that we can make is that this is not a very good way to determine or measure the average caloric value.

But let us analyze the solution a bit further. A 10% error cause the solution to differ dramatically. The value of $x_1$ range from -66 to -26 while the value of $x_2$ range from 99.5 to -49.5. Why so dramatic change for only an error of ±10%? This is an example of a linear system which is ill determined.

**Question 6:** A general equation of line is given by $ax+by = c$ (in standard form). Solve for $y$ and write the equation in slope-intercept form. What is the slope and $y$-intercept in terms of $a,b$ and $c$? Show that the slope does not depend on $c$.

Now let us go back to the actual example. When we consider all possible errors of ±10%, we are only changing the right hand side and the left hand side is the same. For convenience, I copy the table below:

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>18</td>
<td>660</td>
<td>726</td>
<td>594</td>
<td>726</td>
</tr>
<tr>
<td>14</td>
<td>8</td>
<td>480</td>
<td>432</td>
<td>528</td>
<td>528</td>
</tr>
</tbody>
</table>

For the output 660 and 480, we are considering: $12x_1 + 18x_2 = 66014x_1 + 8x_2 = 480$

For the output 726 and 432, we are considering:

$12x_1 + 18x_2 = 72614x_1 + 8x_2 = 432$

Etc

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>18</td>
<td>660</td>
<td>726</td>
<td>594</td>
<td>726</td>
</tr>
<tr>
<td>14</td>
<td>8</td>
<td>480</td>
<td>432</td>
<td>528</td>
<td>528</td>
</tr>
</tbody>
</table>

Now let us focus on two such system of equations:

$12x_1 + 18x_2 = 59414x_1 + 8x_2 = 528$

And

$12x_1 + 18x_2 = 59414x_1 + 8x_2 = 432$

Note that the first equation is common to both.

**Question 7:** Graph the three equations from above on a same set of axis (one equation is common to both – so actually we have three equations). Look at their geometry carefully and what can you conclude?

When two lines are parallel (or almost parallel) then changing a little for the third line can have a drastic effect on the intersection. This is why one of the intersection is (18,22.5) and the other one is (66,-49.5) and they are too apart
from each other. This types of linear system are known as Ill-determined system.

**9. NUMERICAL ANALYSIS PARTIAL PIVOTING AND GAUSS-SEIDEL ITERATION**

In real life you will probably end up with a bigger system of linear equations sometimes in the range of 100’s of equations. Even for a computer to perform Gauss-Jordan elimination can be slow and time consuming. Here we will look at a method of numerical solution known as Gauss-Seidel iteration.

We will start with a simple example of two equations and two unknowns from our previous example:

\[ 12x_1 + 18x_2 = 594 \]
\[ 14x_1 + 8x_2 = 432 \]

First we rearrange the equation so that the diagonal is the largest (this is known as partial pivoting):

\[ 14x_1 + 8x_2 = 432 \]
\[ 12x_1 + 18x_2 = 594 \]

Now we solve the first equation for \( x_1 \), second equation for \( x_2 \) etc. (so we need same number of equation and the variable to apply this method):

\[ x_1 = \frac{(432 - 8x_2)}{14} \]
\[ x_2 = \frac{(594 - 12x_1)}{18} \]

Now start with a guess for the variables. If no guess is available, just choose all to be zero. So let us choose \( x_1 = 0 \) and \( x_2 = 0 \).

Substitute in the first equation and get \( x_1 \):

\[ x_1 = \frac{432}{14} = 30.85 \]

So now we have \( x_1 = 30.85 \) and \( x_2 = 0 \). Substitute this into the second and find \( x_2 \) and then continue with the new value. The hope is that eventually you will get close to the solution. If this is the case we say that the system converge.

**Question 8:** Complete the Gauss-Seidel iteration of our example. Write your result in a table starting from the first iteration. You should do as many as you need before you convince that you get your solution. Does the solution converge to the same solution we got before?

**Question 9:** What happen if we do not perform partial pivoting before Gauss-Seidel iteration? Do the same example above but this time do not do the partial pivoting. Do you get the same result? Is your process a little slower than before?

**10. PROJECT**

Now let us assume in a lake there are two species of fish, say trout and porge. They are eating on four types of insects, say moths, midges, butterfly and roaches (these are represented by \( x_1, x_2, x_3 \) and \( x_4 \) respectively). This time, for each fish you need four equations (why?). Let us say the four equations for trout and porge are given by:

For trout:

\[ 13x_1 + 12x_2 + 7x_3 + 6x_4 = 321 \]
\[ 19x_1 + 6x_2 + 2x_3 + 13x_4 = 283 \]
\[ 3x_1 + 14x_2 + 25x_3 + 8x_4 = 481 \]
24x₁ + 19x₂ + 22x₃ + 15x₄ = 668
Q1: Use Gauss Jordan elimination to solve the system of equations. Feel free to use mathematica for this or any other questions.

Q2: For a possible 5% error write the corresponding all possible outputs and solve them.


Q4: Use Gauss- Seidel iteration after partial pivoting.

Q5: Make some conclusion by analyzing your result. Analyze the sensitivity of the system. Are these ill-determined or well-determined system? Make connection with real life – think about trout and its feeding habit? Which is the favorite food for the trout? Which is least favorite? Do as probable conclusion as you can?

Repeat all the above questions for Porge:

2x₁ + x₂ + 3x₃ + 2x₄ = 60
x₁ + 5x₂ + x₃ + x₄ = 84
6x₁ + 3x₂ + 2x₃ + x₄ = 90
5x₁ + 7x₂ + 2x₃ + x₄ = 136

Q6: Make some final conclusion comparing the feeding habit of trout and porge on that lake.

11. CONCLUSION

The project at the end can also be used as an assessment tools to measure the student learning outcomes which were mentioned in the beginning of the paper. The modulo can easily be modified, changed and altered to fit the various needs of the students and the instructors.

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