Existence-uniqueness of strong solution to a class of quasi-linear damped wave equations with gradient term

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Abstract:
This work investigates global existence and uniqueness of strong solution, corresponding to a class of quasi-linear damped wave equations with gradient term in nonlinear sourcing term

\[ u_{tt} + \alpha u_t = \Delta u + f(x, t, u, \nabla u), (\alpha > 0) \]

in a bounded and \( C^2 \) domain \( \Omega \) in \( \mathbb{R}^n \), where \( f \) satisfying some weak growth restrictions. We obtained the global existence and uniqueness of strong solution \( u \in C^0((0, \infty), H^2(\Omega)) \) by using our previous results [1].

Keywords:
Gradient Term; Strong Solution; Existence and Uniqueness

1. Introduction

We are concerned with the following Neumann or Dirichlet initial-boundary problems for a class of quasi-linear damped wave equations in a bounded and \( C^2 \) domain \( \Omega \) in \( \mathbb{R}^n \):

\[
\begin{cases}
  u_{tt} + \alpha u_t = \Delta u + f(x, t, u, \nabla u) & \text{in } \Omega \times (0, \infty), \\
  u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x) & \text{in } \Omega, \\
  \frac{\partial u}{\partial n} = 0 \text{ or } u = 0 & \text{on } \partial \Omega \times (0, \infty)
\end{cases}
\]

(1.1)

where

\[ u_t = \frac{\partial u}{\partial t}, u_{tt} = \frac{\partial^2 u}{\partial t^2}, \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}, \nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}), x = (x_1, \ldots, x_n). \]

In recent years, there have been extensive studies on well-posedness of the following nonlinear wave equation with general data:

\[
\begin{cases}
  u_{tt} + u_t - u_{xx} = f(u) & \mathbb{R}^+ \times \mathbb{R}^+, \\
  u(0, t) = 0 & t > 0, \\
  (u, u_t)(x, 0) = (u_0, u_1)(x) & x \in \mathbb{R}^+, \quad (1.2)
\end{cases}
\]
where the semilinear term is

\[ f(u) = |u|^p, |u|^{p-1}u, \text{ etc}, \]

with

\[ 1 < p \leq p_c(1,1) \text{ or } p > p_c(1,1), \]

and

\[ p_c(n,k) = 1 + \frac{2}{n+k}. \]

The compatibility conditions are also assumed.

Existence and nonexistence have been developed by Ikehata et al. [2], Ikehata and Ohta [3], etc. and completed in any dimensional space \( \mathbb{R}^n \) by Todorova and Yordanov [4] and Zhang [5]. For the related works, see [6–11] and references therein. Although there are many research papers concerning the initial-boundary value problem in \( \mathbb{R}^n \) for (1.1) (especially studied for the critical exponent \( p \), see [12–15] and references therein), there seems to be little investigations concerning the quasi-linear term \( f \) containing \( \nabla u \) mixed problem for (1.1). This is because in the case it seems difficult to directly construct good basic decay estimates to the quasi-linear problem for (1.1). As compared with the quasi-linear term \( f \) only containing \( u \) case, in the initial-boundary value problem we can rely on good basic decay estimates due to the weak growth restriction for \( f \). Therefore, our main aim is to show the existence and uniqueness with the quasi-linear term \( f \) containing \( \nabla u \) for (1.1).

In [1] we have discussed the abstract damped wave operator equation as follows:

\[
\begin{aligned}
&\frac{d^2u}{dt^2} + k \frac{du}{dt} = G(u), \quad k \geq 0 \\
u(x,0) = \varphi(x), \\
u_t(x,0) = \psi(x).
\end{aligned}
\]

Unfortunately, it is difficulty to classify a class quasi-linear damped wave equations, since the differential dissipativity term is too complex to identify whether has variational property. In [1], we have obtained the existence, uniformly bounded and regularity of solutions by dividing the differential operator \( G(u) \) into two parts, variational and non-variational structure. Obviously, the \( \Delta u \) in Eq. (1.1) has variational structure, while the nonlinear term \( f(x,t,u,\nabla u) \) has non-variational structure. When \( f \) satisfying some weak growth restrictions, we obtained the global existence and uniqueness of strong solution \( u \in C^0((0,\infty),H^2(\Omega)) \). In [16], we have discussed a class of fully nonlinear wave equations with strongly damped terms in a bounded and smooth domain in \( \mathbb{R}^n \), where \( f(x,\Delta u) \) is a given monotone in \( \Delta u \) nonlinearity satisfying some dissipativity and growth restrictions and \( g(x,u,Du,D^2u) \) is in a sense subordinated to \( f(x,\Delta u) \). It is pity that we didn’t obtain the uniqueness of strong solution. Therefore, in this paper we attempted to consider the uniqueness of strong solution to a class damped wave equations which can be divided the nonlinear and dissipativity terms into two parts: variational and non-variational structure. We also attempted to discussed the two kinds of initial-boundary problems.

This paper is organized as follows:
- in Section 2 we recall some preliminary tools, definitions and our previous results;
- in Section 3 we obtained our main results about the mixed problem (1.1).

## 2. Preliminaries and definition

First we introduce a sequence of function spaces:

\[
\begin{aligned}
&X \subset H_2 \subset X_2 \subset X_1 \subset H \\
&X_2 \subset H_1 \subset H.
\end{aligned}
\]
where \( H, H_1, H_2 \) are Hilbert spaces, \( X \) is a linear space, \( X_1, X_2 \) are Banach spaces and all inclusions are dense embeddings. Suppose that

\[
\begin{align*}
L : X & \rightarrow X_1 \\
< Lu, v >_{X_1} & = < u, v >_{H_1}, \quad \forall u, v \in X
\end{align*}
\] (2.2)

In addition, the operator \( L \) has an eigenvalue sequence

\[
L e_k = \lambda_k e_k, (k = 1, 2, \ldots)
\] (2.3)

such that \( \{ e_k \} \subset X \) is the common orthogonal basis of \( H \) and \( H_2 \). We investigate the existence of global solution of the equation (1.3), we need define its weak solution. Firstly, in Banach space \( X \), introduce

\[
L^p((0, T), X) = \{ u : (0, T) \rightarrow X \mid \int_0^T ||u||^p dt < \infty \},
\]

where \( p = (p_1, p_2, \ldots, p_m) \), \( p_i \geq 1 \),

\[
||u||^p = \sum_{k=1}^{m} |u_k|^{p_k},
\]

where \( |\cdot|_k \) is semi-norm in \( X \), and \( ||\cdot||_X = \sum_{i=1}^{m} |\cdot|_i \). Similarly, we can define

\[
W^{1,p}((0, T), X) = \{ u : (0, T) \rightarrow X \mid u, u' \in L^p((0, T), X) \}.
\]

Let \( L^p_{loc}((0, \infty), X) = \{ u(t) \in X \mid u \in L^p((0, T), X), \forall T > 0 \} \).

**Definition 2.1.**
[1] Let \((\varphi, \psi) \in X_2 \times H_1, u \in W^{1,\infty}_{loc}((0, \infty), H_1) \cap L^\infty_{loc}((0, \infty), X_2) \) is called a globally weak solution of (1.3), if for \( \forall \varphi \in X_1, \) it has

\[
< u_t, v >_{H_1} + k < u, v >_{H_1} = \int_0^t < Gu, v >_{H_1} dt + < \varphi, v >_{H_1} + < \psi, v >_{H_1}.
\] (2.4)

**Definition 2.2.**
[1] Let \( Y_1, Y_2 \) be Banach spaces, the solution \( u(t, \varphi, \psi) \) of (1.3) is called uniformly bounded in \( Y_1 \times Y_2 \), if for any bounded domain \( \Omega_1 \times \Omega_2 \subset Y_1 \times Y_2 \), there exists a constant \( C \) which only depends the domain \( \Omega_1 \times \Omega_2 \), such that

\[
||u||_{Y_1} + ||u||_{Y_2} \leq C, \quad \forall (\varphi, \psi) \in \Omega_1 \times \Omega_2 \text{ and } t \geq 0.
\]

**Definition 2.3.**
[17] A mapping \( G : X_2 \rightarrow X_1^* \) is called weakly continuous, if for any sequence \( \{ u_n \} \subset X_2, u_n \rightarrow u_0 \text{ in } X_2, \) it satisfies

\[
\lim_{n \rightarrow \infty} < G(u_n), v > = < G(u_0), v >, \quad \forall v \in X_1.
\]

**Lemma 2.1.**
[17] Let \( H_2, H \) be Hilbert spaces, and \( H_2 \subset H \) be a continuous embedding. Then there exists a orthonormal basis \( \{ e_k \} \) of \( H \), and also is one orthogonal basis of \( H_2 \).
\textbf{Proof.} Let }I : H_2 \to H\text{ be imbedded. According to assume }I\text{ is a linear compact operator, we define the mapping }A : H_2 \to H\text{ as follows}

\[ < Au, v >_{H_2} = < Iu, v >_H = < u, v >_H, \forall v \in H_2. \]

Obviously, }A : H_2 \to H\text{ is linear symmetrical compact operator and positive definite. Therefore, }A\text{ has a complete eigenvalue sequence }\{ \lambda_k \}\text{ and eigenvector sequence }\{ \tilde{e}_k \} \subset H_2\text{ such that}

\[ A\tilde{e}_k = \lambda_k\tilde{e}_k, \quad k = 1, 2, \ldots, \]

and }\{ \tilde{e}_k \}\text{ is orthogonal basis of }H_2\text{. Hence}

\[ < \tilde{e}_i, \tilde{e}_j >_H = < A\tilde{e}_i, \tilde{e}_j >_H = \lambda_i < \tilde{e}_i, \tilde{e}_j >_H = 0, \text{ if } i \neq j. \]

It implies }\{ \tilde{e}_i \}\text{ is also orthogonal sequence of }H\text{. Since }H_2 \subset H\text{ is dense, }\{ \tilde{e}_i \}\text{ is also orthogonal sequence of }H\text{, so }\{ e_i \} = \{ \tilde{e}_i / \| \tilde{e}_i \|_H \}\text{ is norm orthogonal basis of }H\text{. The proof is completed.} \qed

Now, we introduce an important inequality

\textbf{Lemma 2.2.} \cite{Gronwall inequality}

Let }x(t), y(t), z(t)\text{ be real function on }[a, b], \text{ where }x(t) \geq 0, \forall a \leq t \leq b, z(t) \in C[a, b], y(t)\text{ is differentiable on }[a, b]. \text{ If the inequality as follows is hold}

\[ z(t) \leq y(t) + \int_a^t x(\tau)z(\tau)d\tau, \quad a \leq t \leq b, \]

\text{then}

\[ z(t) \leq y(a)e^{\int_a^t x(\tau)d\tau} + \int_a^t e^{\int_\tau^t x(\sigma)d\sigma} dy ds. \]

In Eq. (1.3), suppose that }G = A + B : X_2 \times \mathbb{R}^+ \to X_1^+. \text{ Throughout of this paper, we assume that}

(i) \text{ There exists a function }F \in C^1 : X_2 \to \mathbb{R}^1 \text{ such that}

\[ < Au, Lv > = - < DF(u), v >, \quad \forall u, v \in X \]

(ii) \text{ Function }F\text{ is coercive, i.e.}

\[ F(u) \to \infty \iff \| u \|_{X_2} \to \infty \]

(iii) \text{ }B\text{ as follows}

\[ | < Bu, Lv > | \leq C_1 F(u) + C_2 \| v \|_{H_1}^2, \quad \forall u, v \in X \]

\text{for some }g \in L_\text{loc}^1(0, \infty).

\textbf{Lemma 2.3.} \cite{Gronwall inequality}

Set }G : X_2 \times \mathbb{R}^+ \to X_1^+\text{ is weakly continuous, }\{ \varphi, \psi \} \in X_2 \times H_1, \text{ then we obtain the results as follows}

(1) \text{ If }G = A\text{ satisfies the assumption(i) and (ii), then there exists a globally weak solution of (1.3)}

\[ u \in W_\text{loc}^{1, \infty}((0, \infty), H_1) \cap L_\text{loc}^{\infty}((0, \infty), X_2) \]

and }u\text{ is uniformly bounded in }X_2 \times H_1\text{.}
(2) If $G = A + B$ satisfies the assumption (i),(ii)and (iii), then there exists a globally weak solution of (1.3)

$$u \in W^{1,\infty}_{loc}((0, \infty), H_1) \cap L^{\infty}_{loc}((0, \infty), X_2);$$

(3) Furthermore, if $G = A + B$ satisfies

$$|<Gu, v>| \leq \frac{1}{2} \|v\|^2_F + CF(u) + g(t) \tag{2.10}$$

for some $g \in L^1_{loc}(0, \infty)$, then $u \in W^{2,2}_{loc}((0, \infty), H)$.

### 3. Main Results

In Lemma 2.3, we have obtained our results to a class nonlinear wave operator equations with some very weak conditions. Now, we apply our results to consider the two initial-boundary value problems: Neumann and Dirichlet initial-boundary value problems.

Firstly, we consider the Neumann initial-boundary value problem:

$$\begin{aligned}
&\begin{cases}
\partial_t u + \alpha u = \Delta u + f(x, u, \nabla u) & \text{in } \Omega \times (0, \infty), \\
u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x) & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega \times (0, \infty)
\end{cases}
\end{aligned} \tag{3.1}$$

where $u$ is a scalar function, $\alpha > 0$, $n$ is a unit normal vector of a bounded and $C^2$ domain $\Omega \subset \mathbb{R}^n$. Suppose $f \in C^1$ satisfies

$$\begin{aligned}
&\begin{cases}
|f| + |D_x f| + |D_x g| \leq C(|z| + |\xi|) + g(x, t) \\
|D_x f(x, z, \xi)| + |D_x f(x, z, \xi)| \leq C
\end{cases}
\end{aligned} \tag{3.2}$$

where $g \in L^2((\Omega \times [0, T])$, we have the results as follows:

**Theorem 3.1.**

If the case (3.2) hold, then for any $(\varphi, \psi) \in H^2(\Omega) \times H^1(\Omega)$, $\frac{\partial \varphi}{\partial n} |_{\partial \Omega} = 0$, (3.1) has unique strong solution

$$\begin{aligned}
&\begin{cases}
u \in C^0(0, \infty), H^2(\Omega) \cr
u_t \in C^0(0, \infty), H^1(\Omega) \cr
u_t \in C^0(0, \infty), L^2(\Omega)
\end{cases}
\end{aligned} \tag{3.3}$$

**Remark 3.1.**

Specially, if $f$ is nothing with $\xi$, when $\Omega$, $(\varphi, \psi)$, $f$ are all $C^\infty$, then $u$ is $C^\infty$.

**Proof.** In order to obtain the existence of strong solution to the Neumann initial-boundary value problem (3.1), fix the space of (2.1) as follows:

$$\begin{aligned}
X &= \{ u \in C^\infty(\Omega) | \frac{\partial u}{\partial n} |_{\partial \Omega} = 0 \}, \\
H &= X_1 = L^2(\Omega), H_1 = H^1(\Omega), \\
H_2 &= X_2 = \{ u \in H^2(\Omega) | \frac{\partial u}{\partial n} |_{\partial \Omega} = 0 \}.
\end{aligned}$$
define the inner product of $H_2$

$$<u,v>_{H_2} := \int_{\Omega} (-\Delta u + au)(-\Delta v + av)dx, \ (a > 0)$$

Obviously, the norm of $H_2$ is equal to the norm of $H^2(\Omega)$.

further, define the linear operator $L : X \to X_1$ of (2.2)

$$Lu := -\Delta u + au, \ (a > 0)$$

we easily know the operator satisfies the case (2.2) and (2.3).

Define the mapping $G := A + B : X_2 \to X_1^+$ as follows:

$$<Au,v> := \int_{\Omega} (\Delta u - au)vdx$$

$$<Bu,v> := \int_{\Omega} [f(x,u,\nabla u) + au]vdx$$

and the function $F = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + 2a|\nabla u|^2 + a^2u^2)dx$.

Now, we successively verify the cases (i), (ii), (iii) of Lemma 2.3 for (3.1):

$$\frac{d}{d\tau} F(u + \tau v)|_{\tau=0} = \frac{1}{2} \frac{d}{d\tau} \int_{\Omega} (|\nabla u + \tau \nabla v|^2 + 2a|\nabla u + \tau \nabla v|^2 + a^2(u + \tau v)^2)dx|_{\tau=0}$$

$$= \int_{\Omega} (\Delta u \cdot \nabla v + 2a \nabla u \cdot \nabla v + a^2 u \cdot v)dx$$

$$= <DF(u),v>$$

and

$$<Au,Lv> = -\int_{\Omega} (-\Delta u + au)(-\Delta v + av)dx$$

$$= -\int_{\Omega} (\Delta u \cdot \nabla v + 2a \nabla u \cdot \nabla v + a^2 u \cdot v)dx$$

$$= <DF(u),v>$$

it implies the case (i) of Lemma 2.3, and

$$F(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + 2a|\nabla u|^2 + a^2u^2)dx \to \infty \Leftrightarrow \|u\|_{X_2} \to \infty.$$  

it implies the case (ii) of Lemma 2.3. For $G = A + B : X_2 \to X_1^+$, if $u_n \to u_0$ in $X_2$, for any $v \in X_2$

$$\| G u_n - G u_0, v \|$$

$$= \| (A + B)u_n - (A + B)u_0, v \|$$

$$= \int_{\Omega} (\Delta u_n - \Delta u_0)v + [f(x,u_n,\nabla u_n) - f(x,u_0,\nabla u_0)]vdx$$

$$= \int_{\Omega} (\nabla u_0 - \nabla u_n) \cdot \nabla v + [f(u_n,\nabla u_n) - f(u_n,\nabla u_0) + f(u_n,\nabla u_0) - f(u_0,\nabla u_0)]vdx$$

$$= \int_{\Omega} [D_x f(u_n,\nabla u)(\nabla u_n - \nabla u_0) + D_x f(\nabla u_0)(u_n - u_0)]vdx$$
where $\bar{u}$ is the mean value between $u_n$ and $u_0$, by the case (3.2) and Hölder inequality, from the above formula we have

$$
\int_\Omega [D_x f(u_n, \nabla \bar{u})(\nabla u_n - \nabla u_0) + D_x f(\bar{u}, \nabla u_0)(u_n - u_0)]\,dx
\leq C \int_\Omega \|\nabla u_n - \nabla u_0\| + (u_n - u_0)\,dx
\leq C_1 \left( \int_\Omega \|\nabla u_n - \nabla u_0\|^2\,dx \right)^{\frac{1}{2}} \left( \int_\Omega |v|^2\,dx \right)^{\frac{1}{2}} + C_2 \left( \int_\Omega |u_n - u_0|^2\,dx \right)^{\frac{1}{2}} \left( \int_\Omega |v|^2\,dx \right)^{\frac{1}{2}}
= C_0 \|\nabla u_n - \nabla u_0\|_{L^2} + C \|u_n - u_0\|_{L^2}
\rightarrow 0 \quad (X_2 = H^2(\Omega) \rightarrow L^2(\Omega)).
$$

Consequently,

$$
\| < Gu_n - Gu_0, v > \| \rightarrow 0
$$

that is $G : X_2 \rightarrow X_1^*$ is weakly continuous.

Therefore, on condition that verify the remaining cases (2.9) and (2.10), we can apply the results (2) and (3) of the Lemma 2.3.

By the assumption (3.2)

$$
\| Bu, Lv \| \leq \int_\Omega (f(x, u, \nabla u) + a u)(-\Delta v + av)\,dx
= \int_\Omega \left[ D_x f(x, u, \nabla u)\nabla v + D_x f(x, u, \nabla u)\nabla u \cdot \nabla v + D_x f(x, u, \nabla u)D^2uDv + af(x, u, \nabla u)v + a \nabla u \cdot \nabla v + a^2 uv \right]\,dx
\leq C \int_\Omega \|D^2u\|^2 + |\nabla u|^2 + |u|^2\,dx + C_1 \|v\|_{H^1}^2 + g(t)
\leq C [F(u) + \|v\|_{H^1}^2] + g(t)
$$

it satisfies the case (iii), furthermore,

$$
\| < Gu, v > \| \leq \int_\Omega |-\Delta u + f(x, u, \nabla u)|\,dx
\leq \frac{1}{2} \int_\Omega |v|^2\,dx + \frac{1}{2} \int_\Omega |\Delta u|^2 + |f(x, u, \nabla u)|^2\,dx
\leq \frac{1}{2} \|v\|_{H^1}^2 + CF(u) + g(t)
$$

consequently, the case (2.10) holds.

So we proved the Neumann initial-boundary value problem (3.1) exists a solution $u \in L^\infty((0, T), H^2(\Omega)) \cap W^{1, \infty}((0, T), H^1(\Omega))$ from the conclusions (2) and (3) of Lemma 2.3.

By the theory of operator semigroup [19], the solution of Neumann initial-boundary value problem (3.1) can write as follows:

$$
u = e^{-\frac{q t}{2}}[\cos(-\mathcal{L})^\frac{1}{2} \phi + \frac{\alpha}{2} (-\mathcal{L})^{-\frac{1}{2}} \sin(-\mathcal{L})^\frac{1}{2} \psi + (-\mathcal{L})^{-\frac{1}{2}} \sin(-\mathcal{L})^\frac{1}{2} \psi]
+ \int_0^t e^{\frac{q(t-\tau)}{2}}(-\mathcal{L})^{-\frac{1}{2}} \sin(t-\tau)(-\mathcal{L})^\frac{1}{2} f(x, u, \nabla u)\,d\tau
$$

$$
u_t = -\frac{\alpha}{2} u + e^{-\frac{q t}{2}}[\cos(-\mathcal{L})^\frac{1}{2} \phi + \frac{\alpha}{2} \cos(-\mathcal{L})^\frac{1}{2} \psi + \cos(-\mathcal{L})^\frac{1}{2} \psi]
+ \int_0^t e^{\frac{q(t-\tau)}{2}} \cos(t-\tau)(-\mathcal{L})^\frac{1}{2} f(x, u, \nabla u)\,d\tau
$$
where \( \mathcal{L} = \triangle u + \frac{\partial^2}{\partial t^2} u. \)

\[
\begin{align*}
\mathcal{L} = \triangle u + \frac{\partial^2}{\partial t^2} u. \\
u \in C^0((0,T),H^2(\Omega)), u_t \in C^0((0,T),H^1(\Omega))
\end{align*}
\]

as well as \( u_{tt} \in L^2((0,T) \times \Omega) \) and \( f(x,u,\nabla u) \in C^0((0,T),L^2(\Omega)) \), for any \( u \) of (3.5), we can directly yields from the Neumann initial-boundary value problem (3.1)

\[
u_{tt} \in C^0((0,T),L^2(\Omega)).
\]

Next, we start to prove the uniqueness of strong solution to the Neumann initial-boundary value problem (3.1):

May assume \( u_1,u_2 \in L^\infty((0,T),H^2) \cap W^{1,\infty}((0,T),X_{\frac{1}{2}}) \) are the solutions of equation (3.1), where \( X_{\frac{1}{2}} \) is the \( \frac{1}{2} \) fractional space of fan-shaped operator \( L \), thus by (3.5) we have \( u_i \in C^0((0,T),X_{\frac{1}{2}}) \) \((i = 1, 2)\), and

\[
\|u_1 - u_2\|_{X_{\frac{1}{2}}} = \|(-\mathcal{L})^{\frac{1}{2}}(u_1 - u_2)\|_X \\
\leq C \int_0^T \|f(x,u_1,\nabla u_1) - f(x,u_2,\nabla u_2)\|_X d\tau \\
= C \int_0^T \|f(x,u_1,\nabla u_1) - f(x,u_2,\nabla u_2) + f(x,u_2,\nabla u_1) - f(x,u_2,\nabla u_2)\|_X d\tau \\
\leq C \int_0^T \|D_x f(\eta,\nabla u_1)(u_1 - u_2)\|_X d\tau + \int_0^T \|D_\eta f(u_2,\nabla \eta)(\nabla u_1 - u_2)\|_X d\tau \\
\leq C_1 \int_0^T \|u_1 - u_2\|_X d\tau + C_2 \int_0^T \|\nabla u_1 - \nabla u_2\|_X d\tau
\]

by Poincaré inequality, we immediately obtain

\[
\|u_1 - u_2\|_{X_{\frac{1}{2}}} \\
\leq C_1 \int_0^T \|u_1 - u_2\|_X d\tau + C_2 \int_0^T \|\nabla u_1 - \nabla u_2\|_X d\tau \\
\leq C_1 \int_0^T \|u_1 - u_2\|_{X_{\frac{1}{2}}} d\tau + C_2 \int_0^T \|\nabla u_1 - \nabla u_2\|_{X_{\frac{1}{2}}} d\tau \\
= (C_1 + C_2) \int_0^T \|u_1 - u_2\|_{X_{\frac{1}{2}}} d\tau
\]

finally, by Gronwall inequality we obtain

\[
\|u_1 - u_2\|_{X_{\frac{1}{2}}} \to 0
\]

Consequently, \( u_1 = u_2 \), that is the uniqueness of strong solution is proved.

Finally, if \( f \) is nothing with \( \xi \), when \( \Omega, (\varphi, \psi) \) and \( f \) are all \( C^\infty \), for \( \forall \alpha > 0 \)

\[
u \in C^0([0,T],X_{\alpha}) \cap C^1([0,T],X_{\alpha-1}) \cap C^2([0,T],X_{\alpha-1}) \cap \cdots
\]

Consequently, \( u \in C^\infty([0,T],X_{\alpha}) \), for \( \forall \alpha > 0. \)
Secondly, we consider the Dirichlet initial-boundary value problem.

\[
\begin{cases}
  u_{tt} + au_t = \triangle u + f(x,t,u,\nabla u) & \text{in } \Omega \times (0,\infty), \\
  u(x,0) = \varphi(x), u_t(x,0) = \psi(x) & \text{in } \Omega, \\
  u = 0 & \text{on } \partial\Omega \times (0,\infty)
\end{cases}
\]  

(3.6)

**Theorem 3.2.**
If the case (3.2) hold, then for any \((\varphi,\psi) \in H^2(\Omega) \times H^1(\Omega), \frac{\partial \varphi}{\partial n} = 0,(3.6)\) has unique strong solution

\[
\begin{cases}
  u \in C_0((0,\infty),H^2(\Omega)) \\
  u_t \in C_0((0,\infty),H^1(\Omega)) \\
  u_{tt} \in C_0((0,\infty),L^2(\Omega))
\end{cases}
\]

(3.7)

**Remark 3.2.**
Specially, if \(f\) is nothing with \(\xi\), when \(\Omega, (\varphi,\psi), f\) are all \(C^\infty\), then \(u\) is \(C^\infty\).

**Proof.** Fix the space of the case (2.1) as follows:

\[X = \{u \in C^\infty(\Omega)|u|_{\partial\Omega} = 0, \triangle u|_{\partial\Omega} = 0\}\]

\[X_1 = L^2(\Omega), X_2 = H^2(\Omega) \cap H^1_{0}(\Omega)\]

\[H = L^2(\Omega), H_1 = H^1_{0}(\Omega), H_2 = H^2(\Omega) \cap H^1_{0}(\Omega)\]

define the inner product of \(H_2\)

\[< u, v >_{H_2} := \int_{\Omega} \triangle u \triangle v dx,\]

further, define the linear operator \(L : X \rightarrow X_1\) of (2.2)

\[Lu := -\triangle u + au, (a > 0)\]

define the mapping \(G := A + B : X_2 \rightarrow X_1^*\) as follows:

\[Au := \triangle u\]

\[Bu := f(x,u,\nabla u)\]

and the function \(F : X_2 \rightarrow \mathbb{R}^1\)

\[F(u) = \frac{1}{2} \int_{\Omega} (|\triangle u|^2 + a|\nabla u|^2) dx.\]  

(3.8)

**Remark 3.3.**
The main difference in proof between the Neumann and Dirichlet initial-boundary value problems is that

\[\int_{\Omega} f(x,u,\nabla u) \cdot \triangle v dx \neq - \int_{\Omega} \nabla f(x,u,\nabla u) \cdot \nabla v dx\]
Let \( u \in C^1([0, \infty), X) \), \( u(x, 0) = \varphi \), then

\[
\int_0^t \left< Bu, Lu_t \right> dt = \int_0^t f(x, u, \nabla u)(-\triangle u_t + au_t) dx d\tau
\]

\[
\leq \int_\Omega f(x, u, \nabla u)\triangle u dx + \int_\Omega f(x, \varphi, \nabla \varphi)\triangle \varphi dx
\]

\[
+ \int_0^t \int_\Omega \left| \frac{d}{d\tau} f(x, u, \nabla u) \right| \|\triangle u\|_2 dx dt + a \int_\Omega \int_\Omega |f| |u_t| dx dt
\]

since \( u \in X_2 = H^2(\Omega) \cap H^1_0(\Omega) \), by Poincaré inequality and Young inequality, we have

\[
\int_0^t f(x, u, \nabla u)\triangle u dx
\]

\[
\leq \left( \int_\Omega |\triangle u|^2 dx \right)^{1/2} \left( \int_\Omega |f(x, u, \nabla u)|^2 dx \right)^{1/2}
\]

\[
= \|\triangle u\|_2 \|f\|_2
\]

\[
\leq \frac{1}{4} \|\triangle u\|_2^2 + \|f\|_2^2
\]

combine the two formulas, we have

\[
\int_0^t \left< Bu, Lu_t \right> d\tau \leq \frac{1}{4} \int_\Omega \left( |\triangle u|^2 + 4|f|^2 \right) dx + \int_\Omega |f(x, \varphi, \nabla \varphi)||\triangle \varphi| dx
\]

\[
+ \int_0^t \int_\Omega \left| D_x f \right| |\triangle u| |u_t| + |D_x f| |\triangle u||\nabla u_t| + a|f||u_t| dx dt
\]

\[
\leq \frac{1}{2} F(u) + C + C_1 \int_0^t \int_\Omega \left( |\triangle u|^2 + |\nabla u|^2 + |u_t|^2 \right) dx dt
\]

\[
+ C_2 \int_0^t \int_\Omega \left| \nabla u_t \right|^2 + |u_t|^2 dx dt
\]

it implies that

\[
\int_0^t \left< Bu, Lu_t \right> d\tau \leq \frac{1}{2} F(u) + C \int_0^t |F(u) + \|u_t\|^2_{H_1} d\tau + C_1
\]

therefore, make the case (iii) more weak to

\[
\int_0^t \left< Bu, Lu_t \right> d\tau \leq C \int_0^t |F(u) + \|u_t\|^2_{H_1} d\tau + \alpha F(u) + \beta,
\]

where \( 0 < \alpha < 1, \beta > 0 \), the remain proof is the same as the Neumann initial-boundary value problem, so the theorem (3.2) is completely proved.

\[
\square
\]

References


