Thermal Diffusion in High Intensity Focused Ultrasound

Qian Zuwen*

Institute of Acoustics, Chinese Academy of Sciences, 100190 Beijing, China

*Corresponding author: qianzw@mail.ioa.ac.cn

Abstract:
The Pennes’ bio-heat transfer equation is solved for a point heat source in pulse or CW modes, respectively. These solutions are further applied to the actual set (HIFU) to predict its temperature elevation and cooling processes and several concise analytical expressions are obtained. The results demonstrate that the temperature elevation is proportional reciprocally to the blood perfusion rate and the temperature during the cooling process is reduced exponentially with the increase in blood perfusion rates. It also shows that the higher temperature elevation will be generated when the set works in CW mode than in pulse mode. The appearance and shape of the focal spot in the tissues heated by HIFU set is similar to the isoline of the acoustic intensity as long as the heating dosage is sufficient. Moreover, an acoustic method to measure the blood perfusion rates in vivo is suggested.

Keywords:
High Intensity Focused Ultrasound; Thermal Diffusion; Blood Perfusion Rates

1. INTRODUCTION

In recent years, the high intensity focused ultrasound (HIFU) has been applied to the tumor therapy in clinic. As we know that during the heating by the ultrasonic sets the sound energy is focused in a small area and a part of the acoustic energy is converted into heat through an irreversible thermodynamic process, which results in an observable temperature elevation in the focal region. When the heating dosage is sufficient, the cancer cells can be denaturalized and even killed, to achieve a satisfactory therapeutic efficacy[1–5]. Nowadays, there are two types of working modes in heating sets: CW mode and intermittent (pulse) mode. As for the therapeutic efficacy or demanded dosage is concerned, the temperature elevation is an essential factor. The question is: which mode (pulse or CW) is better?

In order to investigate the thermodynamic behavior of the tissues heated by HIFU set, the Pennes’ bio-heat transfer equation[6] is to be solved,

\[
\frac{\partial T}{\partial t} = \alpha_h \nabla^2 T - \frac{1}{\tau_b} T + \frac{Q}{\rho C},
\]

\[\alpha_h = \frac{\kappa}{\rho C}, \quad \tau_b = \left[\frac{W_b C_b}{\rho C}\right]^{-1},\]

(1) (2)
where $T$ is the temperature elevation, $\rho$ is the density of the tissue, $\kappa$ and $C$ are the thermal conductivity coefficient and the volume specific heat for tissue, respectively, $C_b$ is the specific heat for blood, $W_b$ is the mass flow rate of the perfusion blood, $\tau_0$ is the equivalent cooling time due to the blood perfusion, $Q$ is the rate of heat production per unit volume (for example, the heat is provided by the HIFU heating set).

As is well known, the cooling of the flowing blood will have an obvious effect on the temperature elevation and the therapeutic efficacy in clinic, thus, in the areas where the rate of perfusion is higher, the temperature elevation is smaller and vice versa. In order to research the thermodynamic behavior of the tissues heated by HIFU set during therapy, there were several authors who solved equation (1) analytically[6–8] or numerically[9, 10]. However, the analytical solutions only apply to point heat sources (cf. appendix 1). A few points need to be made clear for all the above mentioned solutions. First of all, these solutions were obtained only in CW mode, not pulse mode; secondly, the temperature distribution resulting from an actual ultrasound source (HIFU) in general is different from a point source; finally, the temperature variation during the heating and cooling processes depends on the constant $\tau_0$. However, since there is almost no data measured in vivo available, these above mentioned authors used different data $\tau_0$ to the numerical computations of them within a range from 50s to 1000s.

In the research conducted by the author of this paper[11], equation (1) was solved where the heating modes are in both CW and pulse modes, and the point source solutions were obtained, respectively. Based on that result, a further investigation for the heating set (HIFU) is carried out in this paper and the concise analytical expressions for the temperature elevation and cooling process in both modes are obtained, respectively, which show us that the temperature elevation in CW mode is always higher than in pulse mode. From these theoretical results, a method to measure the equivalent cooling constant or the blood perfusion rate in vivo is proposed.

1.1 Point source solutions for Pennes’ equation (CW modes)

Let

$$T(r, t) = T_1(r, t)e^{-\frac{t}{\tau_0}}.$$  \hspace{1cm} (3)

Substituting (3) into (1) yields

$$\frac{\partial T_1}{\partial t} = a_0 \nabla^2 T_1 + \frac{Q}{\rho C} e^{-\frac{t}{\tau_0}}.$$  \hspace{1cm} (4)

The initial condition is

$$t = 0, \quad T(r, t) \big|_{t=0} = T_1(r, t)e^{-\frac{t}{\tau_0}} \big|_{t=0} = \varphi(r),$$  \hspace{1cm} (5a)

or

$$T(r, 0) = T_1(r, 0) = \varphi(r)$$  \hspace{1cm} (5b)

The heat source can be written as

$$Q = Q_0(r_0) \delta(r - r_0)U(t), \quad U(t) = \begin{cases} 
0, & t < 0 \\
1, & t > 0 \end{cases}.$$  \hspace{1cm} (6)

where $Q_0(r_0) = Q_0$ is independent of time variable $t$ and $r_0$ is the vector of the location of the point source. Under the initial condition (5b) the solution of equation (4) can be obtained as following[12]:

$$T_1(r, t) = \frac{1}{(2\pi)^{3/2} \kappa^2 \tau_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(r) e^{-\frac{(r_{0x}-r_{x})^2}{2\kappa^2 \tau_0}} d\xi_0 d\eta_0 d\tau_0 + \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(2\sqrt{\pi} \kappa \tau_0) \rho C} \varphi(r) e^{-\frac{(r_{0x}-r_{x})^2}{2\kappa^2 \tau_0}} d\xi_0 d\eta_0 d\tau_0 d\tau.$$  \hspace{1cm} (7)
Since the heating starts in $t > 0$ and $\varphi(r) = 0$ in equation (7), then, one has

$$T_1(r - r_0, t) = \frac{Q_0}{\rho C} e^{\frac{t}{\tau_b}} \int_0^t \frac{1}{2\sqrt{\pi a_h \tau}} e^{-\frac{(r - r_0)^2}{4a_h \tau}} d\tau.$$  

Let

$$\frac{\tau}{\tau_b} = \tau_1.$$  

Substituting it into abovementioned expression yields

$$T(r, t) = T_1(r - r_0, t)e^{-\frac{t}{\tau_b}} = \frac{2Q_0}{\rho C [4\pi a_h]^{\frac{1}{2}} \sqrt{\tau_b}} \int_0^\infty e^{-\frac{1}{4\tau_b} \frac{(r - r_0)^2}{\tau_b} - \frac{1}{2\tau_b} t^2} d\tau,$$

where the subscript of $\tau_1$ was omitted. Equation (8) is just the temperature elevation for CW mode, where $t$ is the heating time. It can be proved in Appendix 1 that the solution in Ref. [7] (which is equation (4) in that paper) can be deduced from equation (8). The cooling integral after heating by CW mode will be indicated in the following (cf. equation (15)).

1.2 Intermittent (pulse) heating modes

The route of the heating for an intermittent (pulse) mode is shown in Figure 1. In initial instant (at the point O in Figure 1) the ultrasound source starts heating from point A to point B, between which the heating maintains a duration $\tau_H$. After that, it starts cooling down for a duration of $\tau_V$ until point C, and the process is repeated periodically. Obviously, when the heating route arrives at the point B, the temperature elevation can be obtained from equation (7) as following

$$T^{(1H)}(r, \tau_H) = \frac{2Q_0}{\rho C [4\pi a_h]^{\frac{1}{2}} \sqrt{\tau_b}} \int_0^\infty e^{-\frac{1}{4\tau_b} \frac{(r - r_0)^2}{\tau_b} - \frac{1}{2\tau_b} t^2} d\tau.$$  

Equation (9) can be obtained when we let the heating period $t$ be $\tau_H$ in equation (8). After then, it starts cooling. In order to calculate the temperature variation during the cooling process, equations (5) and (6) are involved again, where, obviously, the initial temperature (not zero) should be $\varphi(r) = T^{(1H)}(r, \tau_H)$ and $Q=0$ since the heating is stopped. Substituting these into equation (7) then taking the results into (3) yields (cf. Appendix 2).
\begin{align}
T^{(1C)}(r, t) &= T_1^{(1C)}(r - r_0, t) e^{-\frac{t}{\tau_h}} \\
&= \frac{2Q_0}{\rho C} e^{-\frac{t}{\tau_h}} \frac{1}{\sqrt{4\pi a_{th}}} \int_{0}^{\infty} \frac{1}{\sqrt{\frac{a_{th}}{b}}} e^{-\frac{(r - r_0)^2}{4a_{th}b}} \frac{1}{(1 + \frac{r^2}{\tau_h^2})^2} e^{-\frac{t}{\tau_h}} \frac{(r - r_0)^2}{4a_{th}b} \frac{1}{(1 + \frac{r^2}{\tau_h^2})^2} d\tau. 
\end{align}

Subsequently, the second heating starts when the process arrives at point \( C \), where the initial temperature is \( \varphi(r) = T^{(1C)}(r, \tau_H) \). Taking equations (5) -(7) and (A2.6) into account, where

\begin{align}
g_h &= \frac{\tau^2}{4a_{th}\tau_0(1 + \frac{\tau^2}{\tau_h^2})}, \quad g_l = \frac{1}{4a_{th}t},
\end{align}

then one has

\begin{align}
T^{(2H)}(r, \tau_H) &= T^{(1H)}(r, \tau_H) + \\
&= \frac{2Q_0}{\rho C} e^{-\frac{t}{\tau_h}} \frac{1}{\sqrt{4\pi a_{th}}} \int_{0}^{\infty} \frac{1}{\sqrt{\frac{a_{th}}{b}}} e^{-\frac{(r - r_0)^2}{4a_{th}b}} \frac{1}{(1 + \frac{r^2}{\tau_h^2})^2} e^{-\frac{t}{\tau_h}} \frac{(r - r_0)^2}{4a_{th}b} \frac{1}{(1 + \frac{r^2}{\tau_h^2})^2} d\tau.
\end{align}

The second cooling starts when the process arrives at point \( F \), where the initial temperature is \( \varphi(r) = T^{(2H)}(r, \tau_H) \). Taking equations (5) -(7) into account, a similar calculation shown in Appendix 2 can be repeated when

\begin{align}
g_h &= \frac{\tau^2}{4a_{th}\tau_0(1 + \frac{\tau^2}{\tau_h^2})}, \quad g_l = \frac{1}{4a_{th}t},
\end{align}

then one has

\begin{align}
T^{(2C)}(r, t) &= \frac{2Q_0}{\rho C} e^{-\frac{t}{\tau_h}} \frac{1}{\sqrt{4\pi a_{th}}} \left\{ U_0(r - r_0, t) + e^{-\frac{3\tau}{\tau_h}} U_1(r - r_0, t) \right\},
\end{align}

where

\begin{align}
U_n(r - r_0, t) &= \frac{1}{\sqrt{\frac{a_{th}}{b}}} \int_{0}^{\infty} \frac{1}{\sqrt{\frac{a_{th}}{b}}} e^{-\frac{(r - r_0)^2}{4a_{th}b}} \frac{1}{(1 + \frac{r^2}{\tau_h^2})^2} e^{-\frac{t}{\tau_h}} \frac{(r - r_0)^2}{4a_{th}b} \frac{1}{(1 + \frac{r^2}{\tau_h^2})^2} d\tau.
\end{align}

This process can be repeated periodically, finally, the temperature variation after \( N \) cycles of the process can be obtained as following

\begin{align}
T^{(NC)}(r, t) &= \frac{2Q_0}{\rho C} \frac{1}{\sqrt{4\pi a_{th}}} \int_{0}^{\infty} \frac{1}{\sqrt{\frac{a_{th}}{b}}} e^{-\frac{t}{\tau_h}} \sum_{n=0}^{N-1} e^{-\frac{n\tau}{\tau_h}} \frac{a_{th}}{b} U_n(r - r_0, t). 
\end{align}

Obviously, if \( t = 0 \) in equation (13), the temperature elevation resulting from the heating for \( N \) cycles can be obtained as follows

\begin{align}
T^{(NH)}(r, t) &= \frac{2Q_0}{\rho C} \frac{1}{\sqrt{4\pi a_{th}}} \int_{0}^{\infty} \frac{1}{\sqrt{\frac{a_{th}}{b}}} \sum_{n=0}^{N-1} e^{-\frac{n\tau}{\tau_h}} \frac{a_{th}}{b} U_n(r - r_0, 0).
\end{align}

### 1.3 Cooling integral after CW heating

If the CW heating maintains a duration \( t_H \), after that the cooling process starts, during which the temperature variation can be described by equation (10) so long as \( \tau_H = t_H \), then one has

\begin{align}
T^{(C)}(r, t) &= \frac{2Q_0}{\rho C} e^{-\frac{t}{\tau_h}} \frac{1}{\sqrt{4\pi a_{th}}} \int_{0}^{\infty} \frac{1}{\sqrt{\frac{a_{th}}{b}}} e^{-\frac{(r - r_0)^2}{4a_{th}b}} \frac{1}{(1 + \frac{r^2}{\tau_h^2})^2} e^{-\frac{t}{\tau_h}} \frac{(r - r_0)^2}{4a_{th}b} \frac{1}{(1 + \frac{r^2}{\tau_h^2})^2} d\tau.
\end{align}
2. TEMPERATURE ELEVATION AFTER HIFU HEATING

In the above we discussed only the thermal diffusion from a point heat source. In the following, the temperature elevation in tissues heated by HIFU set will be considered, where the region heated by ultrasound is focused at rather than a point but a small area, which can be regarded as numerous point heat sources, so that the distribution function of them can be written as

$$Q(r, t) = \int Q(r_0, t) \delta(r - r_0) dV_0,$$

where $r_0$ and $r$ are the source and field coordinates, respectively, and $dV_0$ is the element volume. Thus, the solution of the temperature elevation corresponding to HIFU heating source as described by Eqs. (16) and (17) should be:

$$T(r, t) = \int T_\delta(r - r_0, r_0, t) dV_0,$$

where the $T_\delta(r - r_0, r_0, t)$ is the point source solution for the relevant situation, for example, it is just the solution (8) for CW mode. Substituting (8) into (18) yields

$$T(r, t) = \frac{1}{\rho C} \int Q_0(r_0) \frac{1}{[4\pi \alpha t_b]^\frac{3}{2}} f(|r_0 - r|, \sqrt{\frac{\tau_0}{t_b}}) dV_0,$$

where

$$f(|r - r_0|, \sqrt{\frac{\tau_0}{t_b}}) = \int_0^\infty e^{-\frac{1}{4} - \frac{(r - r_0)^2}{4\alpha t_b\tau}} d\tau.$$

From the integrals table, one has

$$f(|r - r_0|, \sqrt{\frac{\tau_0}{t_b}}) = \int_0^\infty e^{-\frac{1}{4} - \frac{(r - r_0)^2}{4\alpha t_b\tau}} d\tau = \sqrt{\frac{\pi}{x}} e^{-x},$$

where

$$x = \frac{|r - r_0|}{\sqrt{4\alpha t_b t_b}}.$$

Numerical calculation indicates that $f(0, \sqrt{\frac{\tau_0}{t_b}})$ has an infinite value when $\frac{\alpha t_b}{\tau_0} < \infty$, which means the solution of the point source is a localization field that only offers a contribution of the temperature elevation to its immediate neighbor. On the other hand, by the definition of the dissipation function $\Phi(r_0)$: sound energy is converted to heat through an irreversible thermodynamic process, thus, one has

$$Q_0(r_0) = \Phi(r_0).$$

As is well known, there is a relationship between the sound absorption and dissipation function as following [13–17],

$$\Phi(r_0) = 2\alpha l(r_0),$$
where $\alpha$ is the sound absorption coefficient, $I(r_0)$ is the acoustical intensity. Substituting these results into equation (19) yields

$$T(r,t) = \frac{1}{[4\pi a_0]^2 \sqrt{\tau_b}} \int \frac{4\alpha I(r_0)}{\rho C} f(|r_0 - r|, \sqrt{\frac{\tau_b}{t_H}}) \, d\tau_0.$$  

The actual focused transducer (in HIFU set) is a spherical concave radiator with circular boundaries. Due to the sound diffraction and the manufacture imperfections, the sound energy cannot be converged sharply at a point (geometrical focus) but distributes over an acoustic beam of finite width[18, 19], which means that when $r_0 \to r$ the sound intensity increases gradually but does not approach to infinity. On the other hand, the function $f(|r_0 - r|, \sqrt{\frac{\tau_b}{t_H}})$ does not do so, which approaches infinity when $r_0 \to r$. Thus, $I(r_0)$ can be regarded as a slow-varied function $I(r)$ and taken out from the integral symbol, i.e.

$$T(r,t) = \frac{1}{[4\pi a_0]^2 \sqrt{\tau_b}} \frac{4\alpha I(r)}{\rho C} \int f(|r_0 - r|, \sqrt{\frac{\tau_b}{t}}) \, d\tau_0.$$  

In Appendix 4 this integral was calculated for two heating modes and several concise results were obtained as follows:

1. When the heating duration $t_H > \tau_b$, after a CW heating the cooling integral can be

$$T^{(C)}(r,t) \approx \frac{2\alpha I(r)}{\rho C} - \frac{t}{\tau_b} \tau_b,$$  

approximately.

2. After a pulse heating the cooling integral can be

$$T^{(NC)}(r - r_0, t) \approx \frac{2\alpha I(r)}{\rho C} - \frac{t}{\tau_b} \tau_b - \frac{t H}{\tau_H + \tau_V}.$$  

These results indicate that when the pulse heating arrives at a steady state, the temperature elevation is always below the CW heating and equals a ratio of the heating duration $\tau_H$ to the duration $\tau_H + \tau_V$ of the pulse cycle. On the other hand, the equations (22) and (23) also show us that the temperature elevation is proportional to the sound intensity approximately, which means that the appearance and shape of the focal spots on the tissues heated by HIFU set are similar to the isoline of the acoustic intensity as long as the heating dosage is sufficient.

### 3. TO MEASURE BLOOD PERFUSION RATES ACOUSTICALLY

Based on the equations (22) and (23), a method to measure the equivalent cooling time $\tau_b$ or the blood perfusion rates in vivo can be suggested as the following: heat the tissue in an assigned region by the HIFU set, where the temperature elevation and the cooling process after the heating being stopped will be recorded by a RF thermo-sensor, then one has

$$\tau_b = \frac{t}{\ln T(|r - r_0|, 0) - \ln T(|r - r_0|, t)},$$  

(24)
where \( T(|\mathbf{r} - \mathbf{r}_0|,0) \) is the temperature elevation at the end of heating, \( T(|\mathbf{r} - \mathbf{r}_0|,t) \) is the temperature elevation at the instant \( t \) seconds after the end of heating. By recording the decay process of the temperature elevation, the constant \( \tau_b \) can be obtained from equation (24). Figure 2 shows a temperature record of the cooling process after the tissue was heated up to a steady temperature and then heating was stopped. The last section of the record curve in Figure 2 is the temperature descent curve of the cooling process. By substituting of \( \tau_b \) into equation (2), the blood perfusion rate can be obtained.

4. DISCUSSION

In calculation of the integral (19) an approximation \( Q_0(\mathbf{r}_0) = Q_0(\mathbf{r}) \) or \( I_0(\mathbf{r}_0) = I_0(\mathbf{r}) \) was taken, where one regarded the sound intensity as a slow varying function and took it out from the integral symbol, then one has

\[
T(\mathbf{r},\tau_H) = \frac{4\alpha I(\mathbf{r})}{\rho C} \frac{1}{[4\pi \omega_{th}]^2} \sqrt{\tau_0} \int_0^\infty \int_{\frac{\tau_0}{\tau}} e^{-\frac{(\mathbf{r} - \mathbf{r}_0)^2}{4\omega_{th}\tau}} \tau d\tau d|\mathbf{r}_0 - \mathbf{r}|. \tag{25}
\]

Exactly, this approximation is valid only in the neighborhood of \( \mathbf{r}_0 = \mathbf{r} \). As \( \mathbf{r}_0 \) goes far away from \( \mathbf{r} \), \( I(\mathbf{r}_0) \) may be more and more different from \( I(\mathbf{r}) \), thus, the obtained result may not be equal to equation (19), the error of which will be considered in the following. Obviously, the integrand of integral \( (A4.2) \)

\[
y = |\mathbf{r}_0 - \mathbf{r}|^2 e^{-\frac{|\mathbf{r}_0 - \mathbf{r}|^2}{4\omega_{th}\tau F(\tau,t)}}
\]

has a maximum at the point

\[
|\mathbf{r}_0 - \mathbf{r}|_{m} = x_m = \frac{2\sqrt{\omega_{th} \tau_0 F(\tau,t)}}{\tau}.
\]
By exchanging the sequence of integral and splitting the integral interval \((0, \infty)\) into intervals \(0, \beta x_m\) and \((\beta x_m, \infty)\), the integral carrying out in spatial variables in (25) can be calculated in Appendix 5, which is

\[
\int_{\mathbb{R}^3} e^{-\frac{|r_0 - r|^2}{4x_m^2F(\tau,t)}} r_0\,d^3r = \frac{4\pi}{[F(\tau,t)]^2} \int_0^{1.7x_m} e^{-\frac{|x_0 - r|^2}{4x_m^2F(\tau,t)}} r_0\,d^3r \approx \left(\frac{4\pi a_{th} \tau_b}{\tau}\right)^2.
\]

Since

\[
x = \beta x_m = 2\beta \sqrt{a_{th} \tau_b F(\tau,t)} \frac{\tau}{\tau_b},
\]

where

\[
F(\tau,t) = \left[1 + \frac{t + n(\tau_H + \tau_V)}{\tau_b}\right]^{\frac{1}{2}},
\]

and the variation interval of \(\tau\) is \(\left(\sqrt{\frac{\tau_b}{\tau}}, \infty\right)\), thus, one has

\[
x = \beta x_m = 2\beta \sqrt{a_{th} \tau_b F(\tau,t)} \frac{\tau}{\tau_b} \approx 0.12 \left\{ \begin{array}{ll}
\sqrt{t + n(\tau_H + \tau_V)} \leq \sqrt{t + N(\tau_H + \tau_V)} & , \tau = \infty \\
\tau_b + n(\tau_H + \tau_V) \leq \tau_b + N(\tau_H + \tau_V) & , \tau = \sqrt{\frac{\tau_b}{\tau}}.
\end{array} \right.
\]

where \(a_{th} = 0.0013 \text{cm}^2/\text{sec}\) [20, 21]. In the measurements \(\tau_H + \tau_V = 0.39 \text{sec.}, N = 50\), and \(\tau_H > \tau_b\) when the heating arrived at steady. If \(\tau_b \approx 50 \text{ sec.}\), and \(x = \beta x_m\) is the order of 1cm which is the size of the focal spots of HIFU, in the range where no significant change of sound intensity, and the condition of slow varying function is satisfied, hence the results in equations (22) and (23) are valid approximately.

5. CONCLUSIONS

In this paper equation (1) is solved for both CW and pulse heating modes, and the point source solutions are obtained, respectively. Applying to HIFU heating sets, we have the following results:

(1) The temperature elevation is proportional inversely to the blood perfusion rate and proportional to sound intensity of HIFU set, which means that the appearance and shape of the focal spots in the tissues heated by HIFU set is similar to the isoline of the acoustic intensity as long as the heating dosage is sufficient.

(2) When the heating reaches a steady state, the temperature elevation in pulse mode is always below the CW mode, the former is only a \(\frac{\tau_H}{\tau_H + \tau_V}\) times of the latter.

(3) Based on these results, a method to measure the blood perfusion rate \(\text{in vivo}\) is proposed.
APPENDIX 1

From Equation (7) one has

\[ T(r,t) = T_0(r_0, t_0) = \frac{2Q_0}{pC} \frac{1}{[4\pi a_0]^{\frac{3}{2}}} \int_0^\infty \frac{1}{\sqrt{\pi}} \exp\left(-\frac{1}{\pi} \frac{(r-r_0)^2}{[4\pi a_0]^2} \right) \, d\tau. \] (A1.1)

Let

\[ A^2 = \frac{(r-r_0)^2}{[4\pi a_0]^2} \] (A1.2)

and one has

\[ A \neq 0 \]

\[ d\tau = \frac{1}{A^2 \sqrt{\pi}} \left\{ d(A\tau + \frac{1}{A}) + d(A\tau - \frac{1}{A}) \right\}. \] (A1.3)

Substituting (A1.3) into (A1.1) yields

\[ I_0 = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{1}{\pi} A^2 \tau^2} \, d\tau = \frac{\sqrt{\pi}}{2A} \int_{-A}^{A} e^{-\frac{1}{\pi} x^2} \, dx \]

By splitting the integral interval of the second integral in braces of abovementioned expression into \( [\sqrt{\frac{1}{\pi}} - A, \sqrt{\frac{1}{\pi} + 2A}] \) and \( [\sqrt{\frac{1}{\pi} + 2A}, \infty) \) and \( -\infty, -\sqrt{\frac{1}{\pi}} \), one has

\[ I_0 = \frac{\sqrt{\pi}}{4A} \left\{ e^{2A} \text{erf} \left(A \sqrt{\frac{1}{\pi}} + \sqrt{\frac{1}{\pi} + 2A}\right) + e^{-2A} \left[2 - \text{erf} \left( A \sqrt{\frac{1}{\pi} - A \sqrt{\frac{1}{\pi}} + 2A} \right) \right] \} \] (A1.4)

Substituting Eqs. (A1.2) into (A1.4) yields a result, which is substituted subsequently into (A1.1), finally, one can obtain the Eq. (4) of Ref. [7], where \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} \, du \). (A1.5)

APPENDIX 2

\[ T^{(1C)}(r, t) = T_1^{(1C)}(r - r_0, t) e^{-\frac{r_0^2}{4a_0^2t}}. \] (A2.1)

\[ T_1^{(1C)}(r - r_0, t) = \frac{2Q_0}{pC} \frac{1}{[4\pi a_0]^{\frac{3}{2}}} \int_0^\infty \frac{1}{\sqrt{\pi}} \exp\left(-\frac{1}{\pi} \frac{(r-r_0)^2}{[4\pi a_0]^2} \right) \, d\tau \]

(A2.2)

where

\[ I_1(r - r_0, \tau) = \int_0^\infty \exp\left(-\frac{(r-r_0)^2}{4a_0^2t} - \frac{(r-r_0)^2}{4a_0^2t} \cdot \tau^2 \right) \, dr \] (A2.3)

\[ r_\xi = (\xi, \eta, \zeta), \quad r = (x, y, z), \quad r_0 = (x_0, y_0, z_0), \]

\[ \eta = \frac{\eta_0 + \eta_0 y + \eta_0 z}{\sqrt{\eta_0^2 + \eta_0^2 + \eta_0^2}}, \quad \xi = \frac{\xi_0 + \xi_0 y + \xi_0 z}{\sqrt{\xi_0^2 + \xi_0^2 + \xi_0^2}} + g_{\xi} r_0^2 + g_{\xi} r^2, \]

(A2.4)

\[ g_\eta = \frac{\eta_0^2}{4a_0^2}, \quad g_\eta = \frac{\eta_0^2}{4a_0^2}, \] (A2.5)

\[ I_1(r - r_0, t) = \left( \frac{\pi}{\eta_0^2 + \xi_0^2} \right)^{\frac{1}{2}} \exp\left(-\frac{g_{\eta} r_0^2}{\sqrt{\eta_0^2 + \xi_0^2}} \cdot (r - r_0)^2 \right) \]

\[ = \left( \frac{4a_0^2 \eta_0}{\eta_0^2 + \xi_0^2} \right)^{\frac{1}{2}} \exp\left(-\frac{(r-r_0)^2}{4a_0^2(\eta_0^2 + \xi_0^2)} \cdot \tau^2 \right). \] (A2.6)

Substituting these results into Eqs. (A2.2) and (A2.1), one can obtain Eq. (9).
APPENDIX 3

From the equation (A2.6) in Appendix 2, one can obtain equation (11) as long as

\[ g_h = \frac{\tau^2}{4a_0 \tau_0 (1 + \frac{\tau^2}{\tau_0^2})}, \quad g_l = \frac{1}{4a_0 \tau}. \]

APPENDIX 4

Now, we calculate the integral for temperature elevation in general. Place the origin of the coordinates on the field point, the cooling integral can be rewritten as

\[ T^{(C)}(r,t) = \frac{1}{[4\pi a_0]^2 \sqrt{\pi}} \frac{4\alpha l(r)}{\rho C} e^{-\frac{t}{\tau}} \int_0^\infty e^{-\frac{1}{\tau}} J(\tau,t) d\tau, \quad (A4.1) \]

where

\[ J(\tau,t) = \frac{4\pi}{[F(\tau)]^2} \int_0^\infty e^{-\frac{|r_0 - r|^2}{4a_0 \tau^2}} |r_0 - r|^2 |r_0 - r|^2 d|r_0 - r| \]

\[ = \frac{8\pi a_0 \tau_0}{\tau^2} \int_0^\infty |r_0 - r| d|r_0 - r| e^{-\frac{|r_0 - r|^2}{4a_0 \tau^2}} \]

\[ = \frac{8\pi a_0 \tau_0}{\tau^2} \int_0^\infty e^{-2x^2} dx = \frac{16\pi (a_0 \tau_0)^3 \sqrt{\pi}}{2} \]

\[ = \frac{(4\pi a_0 \tau_0)^3}{\tau^2}. \]

For the cooling integral after a CW heating mode one has

\[ T^{(C)}(r,t) = \frac{1}{[4\pi a_0]^2 \sqrt{\pi}} \frac{4\alpha l(r)}{\rho C} e^{-\frac{t}{\tau}} \int_0^\infty e^{-\frac{1}{\tau}} J(\tau,t) d\tau \]

\[ = \frac{1}{[4\pi a_0]^2 \sqrt{\pi}} \frac{4\alpha l(r)}{\rho C} e^{-\frac{t}{\tau}} \int_0^\infty e^{-\frac{1}{\tau}} \left(4\pi a_0 \tau_0 \right) \frac{3}{\tau^3} d\tau \]

\[ = \frac{2\alpha l(r)}{\rho C} \tau_0 \left[ 1 - e^{\frac{-2t}{\tau}} \right] e^{-\frac{t}{\tau}}, \]

When \( t_H < \tau_0 \), then we have

\[ T(r,t) \approx \frac{2\alpha l(r)}{\rho C} t_H e^{-\frac{t}{\tau}}. \]

If \( t_H > \tau_0 \), one has

\[ T(r,t) \approx \frac{2\alpha l(r)}{\rho C} \tau_0 \frac{1}{\tau} e^{-\frac{t}{\tau}}. \]

For the cooling integral after a pulse heating mode one has

\[ T^{(MC)}(r,t) = \int_{[t]} T_0(r - r_0, t) dr_0 \]

\[ = 4\pi \int_0^\infty T_0(r - r_0, t) |r_0 - r| dr_0, \]

where

\[ T_0(r - r_0, t) = T^{(NC)}(r,t) = \frac{2\alpha l(r)}{\rho C} \int_0^\infty e^{-\frac{t}{\tau}} \sum_{n=0}^{\infty} e^{-\frac{3n+2\tau}{\tau_{P}}} U_n(r - r_0, t) \]

and

\[ U_n(r - r_0, t) = \int_{\frac{\tau}{\sqrt{\pi}}}^\infty \frac{1}{[F(\tau,t)]^2} e^{-\frac{1}{\tau^2} \frac{[r_0 - r]^2}{4a_0 \tau^2}} d\tau, \]

where

\[ F(\tau,t) = [1 + \frac{t + n(\tau_H + \tau_P)}{\tau_b \tau^2}]. \]
From (A.4.2) one similarly has

\[ T^{(MC)}(r, t) = \frac{2\alpha I(r)}{\rho C} - \frac{1}{\tau_b} \left[ 1 - e^{-\frac{3\tau}{\tau_b}} \right] e^{-\frac{t}{\tau_b}} \left[ 1 - e^{-\frac{3\tau}{\tau_b}} \right]. \]

In general

\[ \frac{\tau_H + \tau_V}{\tau_b} \ll 1, \]
then one has

\[ T^{(MC)}(r, t) = \frac{2\alpha I(r)}{\rho C} \tau_b e^{-\frac{t}{\tau_b}} \left[ 1 - e^{-\frac{3\tau}{\tau_b}} \right]. \]

When the heating reaches the steady state

\[ N \frac{\tau_H + \tau_V}{\tau_b} > 1, \]

one has

\[ T^{(MC)}(r, t) = \frac{2\alpha I(r)}{\rho C} \tau_b e^{-\frac{t}{\tau_b}} \left[ 1 - e^{-\frac{3\tau}{\tau_b}} \right]. \]

**APPENDIX 5**

The integrand of the integral (A4.2)

\[ y = |r_0 - r|^2 e^{-\frac{|r_0 - r|^2}{4\alpha a \tau_b f(t)}}, \]

has a maximum, of which the coordinate is

\[ |r_0 - r|_m = x_m = \frac{2\sqrt{2} a \tau_b F(t)}{\tau}, \]

(A5.1)
then one has

\[ J(r, t) = \frac{4\pi}{[F(t)]^2} \int_0^\infty e^{-\frac{|r_0 - r|^2}{4\alpha a \tau_b f(t)}} |r_0 - r|^2 d|r_0 - r| \]

\[ = \frac{4\pi}{[F(t)]^2} \left\{ \left[ \int_0^\infty e^{-x_0^2} \right] \left[ \int_0^\infty e^{-x_0^2} \right] e^{-\frac{|r_0 - r|^2}{2\tau_b}} |r_0 - r|^2 d|r_0 - r| \right\} \]

(A5.2)
\[ = -\frac{4\pi a \tau_b}{\tau^3} \left\{ \left[ 2\beta \right] e^{-\beta^2} - \text{erf}(\beta) \right\} - \text{erfc}(\beta) \]

where \( \text{erf}(\beta) \) and \( \text{erfc}(\beta) \) are the error function and complementary error function, respectively. When \( \beta \geq 1.7 \), the quantities in the braces approach to 1, then we have

\[ J(r, t) = \frac{4\pi}{[F(t)]^2} \int_0^\infty e^{-\frac{|r_0 - r|^2}{4\alpha a \tau_b f(t)}} |r_0 - r|^2 d|r_0 - r| \approx \frac{(4\pi a \alpha \tau_b)^2}{\tau^3}. \]

Substituting these results into the heating integral and cooling integral yields:

1. Since

\[ F(t) = 1 + \frac{t}{\tau}, \]
the cooling integral (after a CW heating) can be

$$T^{(C)}(r, t) = \frac{4\alpha l(r)}{\rho c^2} \left( \frac{2\pi \alpha n \delta_r}{\rho c^2} \right)^{\frac{3}{2}} \int_0^\infty e^{-\frac{1}{\tau^2}} \frac{1}{\tau^3} d\tau$$

or

$$= \frac{4\alpha l(r)}{\rho c^2} e^{-\frac{t}{\tau_0}} \left[ 1 - e^{-\frac{t}{\tau_0}} \right].$$

(5.3)

1. From Eqs. (14)-(16), the cooling integral (after a pulse heating) can be

$$T^{(MC)}(r, t) = \int_0^\infty T_0(r - r_0, t) d\tau_0 = 4\pi \int_0^\infty T_0(r - r_0, t) d|\tau| = \frac{4\alpha l(r)}{\rho c^2} \left( \frac{2\pi \alpha n \delta_r}{\rho c^2} \right)^{\frac{3}{2}} \int_0^\infty e^{-\frac{1}{\tau^2}} \frac{1}{\tau^3} d\tau$$

where

$$T_0(r - r_0, t) = T^{(NC)}(r, t) = \frac{4\alpha l(r)}{\rho c^2} \left( \frac{2\pi \alpha n \delta_r}{\rho c^2} \right)^{\frac{3}{2}} e^{-\frac{1}{\tau^2}} \sum_{n=0}^{N-1} e^{-n \frac{2\pi}{\delta_r}} U_n(r - r_0, t)$$

and

$$U_n(r - r_0, t) = \int_0^\infty \frac{1}{\sqrt{\pi \tau}} e^\frac{t}{\tau} \left[ 1 - e^{-\frac{t}{\tau}} \right] d\tau$$

(A5.6)

Obviously, where

$$F(\tau, t) = \left[ 1 + \frac{\tau}{\tau_0} \right].$$

(A5.7)

One has

$$J(r, t) = \frac{4\pi}{\sqrt{1 + \frac{t}{\tau_0}}} \int_0^\infty e^{-\frac{1}{\tau^2}} \left[ 1 - e^{-\frac{t}{\tau}} \right] d\tau$$

(A5.8)

Substituting Eqs. (A5.5)-(A5.7) into (A5.4), one has

$$T^{(MC)}(r - r_0, t) = \frac{4\alpha l(r) \tau_0}{\rho c^2} e^{-\frac{t}{\tau_0}} \sum_{n=0}^{N-1} e^{-n \frac{2\pi}{\delta_r}} \int_0^\infty \frac{1}{\sqrt{\pi \tau}} e^{-\frac{1}{\tau^2}} d\tau$$

(A5.9)

where one assumes \( \frac{2\pi}{\delta_r} < 1 \). If \( N \frac{2\pi}{\delta_r} > 1 \), we have approximately

$$T^{(NC)}(r - r_0, t) \approx \frac{2\alpha l(r) \tau_0}{\rho c^2} e^{-\frac{t}{\tau_0}} \sum_{n=1}^{N-1} e^{-n \frac{2\pi}{\delta_r}} \tau_b$$

References


About This Journal

APHY is an open access journal published by Scientific Online Publishing. This journal focus on the following scopes (but not limited to):

- Biophysics and Biosensors
- Instrumentation and Metrology
- Magnetism
- Nanomaterials and Nanotechnologies
- Optical Manipulation
- Photonics
- Physics and Mechanics of Fluids, Microfluidics
- Physics of Energy Transfer, Conversion and Storage
- Physics of Organic Materials and Devices
- Semiconductors and Devices
- Singular Optics
- Spectroscopy
- Spintronics
- Superconductivity
- Surfaces and Interfaces
- Thin Films

Welcome to submit your original manuscripts to us. For more information, please visit our website: http://www.scipublish.com/journals/APHY/

You can click the bellows to follow us:

- Facebook: https://www.facebook.com/scipublish
- Twitter: https://twitter.com/scionlinepub
- LinkedIn: https://www.linkedin.com/company/scientific-online-publishing-usa
- Google+: https://google.com/+ScipublishSOP

SOP welcomes authors to contribute their research outcomes under the following rules:

- Although glad to publish all original and new research achievements, SOP can’t bear any misbehavior: plagiarism, forgery or manipulation of experimental data.
- As an international publisher, SOP highly values different cultures and adopts cautious attitude towards religion, politics, race, war and ethics.
- SOP helps to propagate scientific results but shares no responsibility of any legal risks or harmful effects caused by article along with the authors.
- SOP maintains the strictest peer review, but holds a neutral attitude for all the published articles.
- SOP is an open platform, waiting for senior experts serving on the editorial boards to advance the progress of research together.